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Exact Model for a Gaseous Regular Bouncing Sphere in General Relativity

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An exact model for a relativistic "gaseous" sphere (i.e., one whose density ρ vanishes at the outer boundary of the nonstatic sphere together with the pressure p) is given. The model has a bounce: The collapsing sphere comes momentarily to rest when the boundary is still outside the Schwarzschild radius of the matter sphere, then there is a macroscopic bounce, and the matter of the expanding sphere spreads all over the universe. This bouncing solution of Einstein's field equations is physically valid at any moment, i.e., the pressure and the density are positive inside the fluid sphere, and their respective gradients are negative. The mass function is positive, and the circumference is an increasing function of radial coordinate. This solution may represent at easily surveyable model for a supernova explosion where the explosion is so violent that no remnant whatsoever is left.

1. INTRODUCTION

The question of the final, relativistic stage in the evolution a star is now of great interest. The supernova that went off in the Large Magellanic Cloud gives us a unique opportunity to follow a relativistic phenomenon in great detail. Since we cannot claim to have reliable knowledge of the extreme physical conditions under which matter exists at the bounce of a supernova explosion, it is the purpose of this paper to give a simple exact model that may represent such an explosion without having to specify which equation of state the matter obeys.

The progenitor of the supernova SN 1987A is almost certainly the B supergiant Sanduleak-69202. This was certainly a gaseous sphere, and accordingly we construct a nonstatic model where both the matter density and the pressure drop to zero at the surface of the sphere.

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To have a realistic model for stellar collapse and supernova explosions one has, of course, to use numerical calculations. But workers in this field consider two different mechanisms for expelling the star's outer layers:

1. The gravitational energy is released in the form of neutrinos, and these neutrinos transfer momentum to the outer layers and blow them off.
2. The core becomes extremely stiff when the neutrons become degenerate. The outer layers collapse onto this core, and there is a macroscopic bounce of these layers.

We try to keep as close to reality as possible, and we thus choose our model in such a way that the gaseous sphere bounces when its outer boundary is still outside its Schwarzschild radius, and such that its surface area reaches a minimum at this moment.

However, the procedure we follow to obtain our simple, exact model starts with a preassigned form of the spherically symmetric line element (McVittie metric) and requires the distribution of material to be a perfect fluid. Einstein's field equations are then used "in reverse" to obtain the pressure p and the density ρ. But it should be emphasized that this method yields unphysical pressure-density configurations more frequently than physical ones. The model must therefore be carefully checked to see if it is of physical relevance. The pressure and the density should then be positive inside the boundary of the sphere and should be decreasing outward. We show that for our model the pressure and the density have the desired behavior. Moreover, the mass function is positive, and the circumference is an increasing function of radial coordinate. The model has no singularity, and the contraction period is finite.

The model constructed in this paper rests heavily on the results obtained in a previous paper (Knutsen, 1985), and to save space I shall not outline those results in detail, but assume that the reader is rather familiar with that paper.

2. THE GENERALIZED McVITTIE METRIC

We choose the line element in the following way:

$$ds^2 = y^2\,dt^2 - S^2\,e^{\eta}[dr^2 + f^2(d\theta^2 + \sin^2\theta\,d\rho^2)] \tag{1}$$

The scale function S is here a function of t alone, and f is a function of the comoving radial coordinate r alone. Further, y and η are functions of a variable z defined by

$$e^z = Q/S^n \tag{2}$$

where Q is another function of r and n is a constant. The McV metric (McVittie, 1967) is the special case where the constant n is put equal to unity.

For a perfect fluid we now obtain from $T_1^4 = 0$ (T_μ^ν denotes the energy-momentum tensor)

$$y = 1 - \tfrac{1}{2} n \, d\eta / dz \tag{3}$$

The isotropy condition breaks up into the following three differential equations, which may be solved independently (Nariai, 1968):

$$\frac{Q''}{Q} - \frac{Q'f'}{Qf} = a\left(\frac{Q'}{Q}\right)^2 \tag{4}$$

$$\frac{f''}{f} - \frac{f'^2}{f^2} + \frac{1}{f^2} = b\left(\frac{Q'}{Q}\right)^2 \tag{5}$$

$$\frac{d^2y}{dz^2} + \left[a - 1 + \frac{1}{n}(y - 2)\right]\frac{dy}{dz} + \left[\frac{1}{n}(a-1)(1-y) - \frac{1}{n^2}(1-y)^2 + b\right]y = 0 \tag{6}$$

where a prime means differentiation with respect to r, and a and b are arbitrary constants.

We thus have four differential equations to find the four unknown functions y, η, f, Q. The scale function S is found by fitting the internal solution to an external vacuum Schwarzschild solution, i.e., the pressure must be put equal to zero at the boundary.

The general expressions for the density and the pressure have been given in a previous paper (Knutsen, 1983), and there it was also shown that the junction condition at the boundary yields an ordinary differential equation of first order for $\dot{S}^2$ (the dot denotes differentiation with respect to t). When that differential equation is solved, it will thus contain an arbitrary integration constant. However, if we in addition demand the fluid to be gaseous, i.e., we demand the density to vanish at the boundary, that integration constant is in fact specified. The reason for this is that conservation of energy, i.e.,

$$T_{4;\nu}^\nu = 0 \tag{7}$$

[a semicolon denotes covariant differentiation, and for the McV metric equation (7) reads $\dot{\rho} = -3y(\dot{S}/S)(\rho + p)$] yields that for a nonstatic gaseous sphere the pressure must also drop to zero at the boundary.

The pressure gradient is found from the law of conservation of linear momentum, i.e.,

$$T_{1;\nu}^\nu = 0 \tag{8}$$

and for the McV metric it reads

$$p' = -\frac{Q'}{yQ}\frac{dy}{dz}(\rho + p) \tag{9}$$

We have previously (Knutsen, 1983) worked out the expression for the density gradient, and we have also given a simple proof (Knutsen, 1987) which yields the necessary and sufficient conditions for the pressure to be well-behaved for gaseous spheres with negative density gradient. For the generalized McV metric this means that we must prove that $\rho' < 0$ and $(Q'/yQ)\, dy/dz > 0$ to have a gaseous model with negative pressure gradient and thus positive pressure.

3. THE BOUNCING MODEL

We now choose $a = 1 + 2/n$ and $b = 0$, i.e., we follow Section 6.1 in our previous work (Knutsen, 1985). We thus have

$$f = \sinh r \tag{10}$$

where α and β are constants, and

$$Q^{-2/n} = \frac{1}{\alpha} \cosh r + 1 - \beta \tag{11}$$

$$y = \frac{S - Z}{S + Z} \tag{12}$$

where

$$Z = kQ^{1/n} \tag{13}$$

and k is a positive constant.

Further, we have

$$e^{-\eta} = e^{\varepsilon} S^4/(S + Z)^4 \tag{14}$$

where ε is an arbitrary integration constant. The differential equation for the scale function S now reads

$$\dot{S}^2 = e^{\varepsilon} \frac{S^4}{(S + k)^5} (S - S_B) \tag{15}$$

where

$$S_B = k\left[\frac{1}{\alpha^2} - (1 - \beta)^2\right] \tag{16}$$

This equation gives the qualitative time development, and we also have a consistency condition which must be fulfilled, i.e., we must have $\dot{S}^2 \geq 0$.

4. REGULARITY

In our previous work it was found that to have a negative density gradient it was necessary and sufficient that the constants α and β be chosen in the following way:

$$\alpha > 0, \qquad \frac{1}{\alpha} < \beta < 1 + \frac{1}{\alpha} \tag{17}$$

Since y must be positive, we also demend

$$S > Z_{\max} = k(1 + 1/\alpha - \beta)^{-1/2} \tag{18}$$

From equation (11) we now obtain

$$Q'/n < 0 \tag{19}$$

and we also have

$$\frac{dy}{dz} = -\frac{2}{n} \frac{SZ}{(S+Z)^2} \tag{20}$$

Thus, it must be the case that

$$\frac{Q'}{yQ} \frac{dy}{dz} > 0 \tag{21}$$

and according to our previous statement we can now safely conclude that *the pressure and the density and their respective gradients are all well-behaved.* It should be noted that this statement is valid at any moment, and we do not have to restrict its validity to a certain time interval as we did in our previous work (Knutsen, 1985). From that work we also conclude that the mass function m is positive, and the "physical" radius $R = fS\,e^{\eta/2}$ is an increasing function of radial coordinate. It is also immediately seen that the mass function m and the "physical" radius R are well-behaved at the origin, i.e., they both drop to zero at the center of the gaseous sphere.

As stated by Misner and Sharp (1964), we must also have Lorentz-Minkowski geometry at the origin, and this condition now reads

$$\left[f' + (1-y)\frac{Q'f}{nQ}\right] = 1 \qquad \text{for} \quad r = 0 \tag{22}$$

From equation (10) it is seen that this is fulfilled. It is also easily checked that p, p', ρ, and ρ' are all finite at the origin. To have a bouncing solution we must also demand

$$S_B > Z_{\max} \tag{23}$$

and from equations (16) and (18) it is seen that this condition may be written:

$$(1 + 1/\alpha - \beta)^{-3/2} \leq 1/\alpha + \beta - 1 \tag{24}$$

It must also be the case that the boundary is still outside the Schwarzschild radius of the gaseous sphere at the moment of the bounce, i.e.,

$$R_{\text{boundary}}(S_B) > 2m_{\text{boundary}} \tag{25}$$

A little calculation then shows that this condition reads

$$\{1-[(1-\beta)^2-1/\alpha^2]\}^2\alpha^2 > (\alpha^2\beta^2-1)^2[1/\alpha^2-(1-\beta)^2] \tag{26}$$

It is now trivial to check that all our conditions (17), (24), and (26) are fulfilled if we choose, for example, $(\alpha, \beta) = (1/2, 5/2)$ or $(\alpha, \beta) = (1/4, 9/2)$.

From equation (15) we also find that we have

$$\ddot{R}_{\text{boundary}}(S_B) = e^{\varepsilon/2}\frac{f_{\text{boundary}}(S_B-k)S_B^2}{2(S_B+k)^4} > 0 \tag{27}$$

Hence, we can safely conclude that our model represents a truly bouncing sphere and not an asymptotically contracting fluid sphere. This is also confirmed if we calculate the length T of the contraction period. Using equation (15), we find

$$T = e^{-\varepsilon/2}\int_{S_{\text{initial}}}^{S_B}(S+k)^{5/2}\frac{dS}{S^2(S-S_B)^{1/2}} \tag{28}$$

and it is seen that T is finite.

5. CONCLUSION

I have given an exact model for a relativistic *gaseous* sphere. The motion starts with the matter already in motion, and contracts to a state of rest when its boundary is still outside its Schwarzschild radius. The sphere then starts expanding and its matter is spread all over the universe. The sphere is also regular everywhere at any moment. I put this solution forward with the tentative presumption that it is the first model of this kind.

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